

Smoothness of the Orlicz Norm in Musielak–Orlicz Function Spaces

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Abstract

In this paper, we present a characterization of support functionals and smooth points in L_0^Φ , the Musielak–Orlicz space equipped with the Orlicz norm. As a result, criterion for the smoothness of L_0^Φ is also obtained. Some expressions involving the norms of functionals in $(L_0^\Phi)^*$, the topological dual of L_0^Φ , are proved for arbitrary Musielak–Orlicz functions.

Keywords: Musielak–Orlicz spaces, Orlicz norm, smoothness.

1 Introduction

Characterization of support functionals and smooth points, as well as criterion for the smoothness of Musielak–Orlicz (function) spaces equipped with the Orlicz norm, which we denote by L_0^Φ , are already known [7] when the Musielak–Orlicz function Φ is finite-valued and $\Phi(t, u)/u \rightarrow \infty$ as $u \rightarrow \infty$ for μ -a.e. $t \in T$. In this paper, we show these results for arbitrary Musielak–Orlicz functions, which can take values in the extended real numbers. The proofs follow the main lines of the paper [7], with improvements. For instance, we have neither used the Bishop–Phelps Theorem [9], nor the concept of measurable selectors [1]. (As a consequence, see the functions u_* and u^* constructed in Remark 20.) To find these characterizations, some expressions involving the norms of functionals in $(L_0^\Phi)^*$, the topological dual of L_0^Φ , are extended to arbitrary Musielak–Orlicz functions. In the proof of these extensions, a strategy consisted of taking a sequence of finite-valued Musielak–Orlicz functions converging upward to an arbitrary Musielak–Orlicz function (see Lemma 2). Next we present some definitions and results related to Musielak–Orlicz spaces [10, 8].

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Let (T, Σ, μ) be a non-atomic, σ -finite measure space. We say that $\Phi: T \times [0, \infty] \rightarrow [0, \infty]$ is a *Musielak–Orlicz function* if, for μ -a.e. $t \in T$,

- (i) $\Phi(t, \cdot)$ is convex and lower semi-continuous,
- (ii) $\Phi(t, 0) = \lim_{u \downarrow 0} \Phi(t, u) = 0$ and $\Phi(t, \infty) = \infty$,
- (iii) $\Phi(\cdot, u)$ is measurable for all $u \geq 0$.

The *complementary function* $\Phi^*: T \times [0, \infty] \rightarrow [0, \infty]$ to a Musielak–Orlicz function Φ is defined by

$$\Phi^*(t, v) = \sup_{u > 0} (uv - \Phi(t, u)), \quad \text{for all } v \geq 0. \quad (1)$$

It can be verified that Φ^* is a Musielak–Orlicz function. Given any Musielak–Orlicz function Φ , we denote $\partial\Phi(t, u) = [\Phi'_-(t, u), \Phi'_+(t, u)]$, where $\Phi'_-(t, u)$ and $\Phi'_+(t, u)$ are the left- and right-derivatives of $\Phi(t, \cdot)$ at any $u \geq 0$. The functions Φ and Φ^* satisfy the *Young's inequality*

$$uv \leq \Phi(t, u) + \Phi^*(t, v), \quad \text{for all } u, v \geq 0, \quad (2)$$

which reduces to an equality when $v \in \partial\Phi(t, u)$ if u is given, or when $u \in \partial\Phi^*(t, v)$ if v is given. We also define the functions $a_\Phi(t) = \sup\{u \geq 0 : \Phi(t, u) = 0\}$ and $b_\Phi(t) = \sup\{u \geq 0 : \Phi(t, u) < \infty\}$.

Let L^0 denote the space of all real-valued measurable functions on T , with equality μ -a.e. Given a Musielak–Orlicz function Φ , we define the functional

$$I_\Phi(u) = \int_T \Phi(t, |u(t)|) d\mu, \quad \text{for any } u \in L^0. \quad (3)$$

The *Musielak–Orlicz (function) space*, *Musielak–Orlicz (function) class*, and *(function) space of finite elements* are given by

$$\begin{aligned} L^\Phi &= \{u \in L^0 : I_\Phi(\lambda u) < \infty \text{ for some } \lambda > 0\}, \\ \tilde{L}^\Phi &= \{u \in L^0 : I_\Phi(u) < \infty\}, \end{aligned}$$

and

$$E^\Phi = \{u \in L^0 : I_\Phi(\lambda u) < \infty \text{ for all } \lambda > 0\},$$

respectively. Clearly, $E^\Phi \subseteq \tilde{L}^\Phi \subseteq L^\Phi$. The Musielak–Orlicz space L^Φ can be viewed as the smallest vector subspace of L^0 that contains \tilde{L}^Φ , and E^Φ is the largest vector subspace of L^0 that is contained in \tilde{L}^Φ .

The Musielak–Orlicz space L^Φ is a Banach space when it is endowed with any of the norms:

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 : I_\Phi \left(\frac{u}{\lambda} \right) \leq 1 \right\}, \quad (4)$$

$$\|u\|_{\Phi,0} = \sup \left\{ \left| \int_T u v d\mu \right| : v \in \tilde{L}^{\Phi*} \text{ and } I_{\Phi*}(v) \leq 1 \right\}, \quad (5)$$

and

$$\|u\|_{\Phi,A} = \inf_{k>0} \frac{1}{k} (1 + I_\Phi(ku)), \quad (6)$$

which are called the *Luxemburg*, *Orlicz* and *Amemiya norms*, respectively. The Musielak–Orlicz space equipped with the Orlicz norm is denoted by L_0^Φ . The Luxemburg and Orlicz norms are equivalent and are related by the inequalities $\|u\|_\Phi \leq \|u\|_{\Phi,0} \leq 2\|u\|_\Phi$, for any $u \in L^\Phi$. In addition, as shown in [6, 4], the Orlicz and Amemiya norms coincides, i.e., $\|u\|_{\Phi,0} = \|u\|_{\Phi,A}$, for all $u \in L^\Phi$. The Amemiya norm is a special case of the p -Amemiya norm for $p = 1$. For more details on p -Amemiya norms we refer to [3].

For any $u \in L^\Phi$, we denote by $K(u)$ the set of all $k > 0$ for which the infimum in (6) is attained. If $I_{\Phi*}(b_{\Phi*} \chi_{\text{supp } u}) > 1$, where $\text{supp } u = \{t \in T : |u(t)| > 0\}$, and we denote

$$k_u^* = \inf \{k > 0 : I_{\Phi*}(\Phi'_+(t, |ku(t)|)) \geq 1\},$$

$$k_u^{**} = \sup \{k > 0 : I_{\Phi*}(\Phi'_+(t, |ku(t)|)) \leq 1\},$$

then $0 < k_u^* \leq k_u^{**} < \infty$, and $\|u\|_{\Phi,0} = \frac{1}{k}(1 + I_\Phi(ku))$ if and only if $k \in [k_u^*, k_u^{**}]$. If $I_{\Phi*}(b_{\Phi*} \chi_{\text{supp } u}) \leq 1$, then $\|u\|_{\Phi,0} = \int_T |u| b_{\Phi*} d\mu$.

If we can find a non-negative function $f \in \tilde{L}^\Phi$ and a constant $K > 0$ such that

$$\Phi(t, 2u) \leq K\Phi(t, u), \quad \text{for all } u \geq f(t), \quad (7)$$

then we say that Φ satisfies the Δ_2 -condition, or belongs to the Δ_2 -class (denoted by $\Phi \in \Delta_2$). The spaces E^Φ and L^Φ coincide when Φ satisfies the Δ_2 -condition. On the other hand, if Φ is finite-valued and does not satisfy the Δ_2 -condition, then the Musielak–Orlicz class \tilde{L}^Φ is not open and its interior coincides with

$$B_0(E^\Phi, 1) = \{u \in L^\Phi : \inf_{v \in E^\Phi} \|u - v\|_{\Phi,0} < 1\},$$

or, equivalently, $B_0(E^\Phi, 1) \subsetneq \tilde{L}^\Phi \subsetneq \overline{B_0(E^\Phi, 1)}$.

Musielak–Orlicz spaces are endowed with the structure of Banach lattices [2]. This property can be used in a more refined analysis of the (topological) dual space of L^Φ ,

which is denoted by $(L^\Phi)^*$. Fixed $v \in L^{\Phi^*}$, the expression

$$f_v(u) := \int_T uv d\mu, \quad \text{for all } u \in L^\Phi, \quad (8)$$

defines a functional in $(L^\Phi)^*$. A functional that can be written in the form (8) is said to be *order continuous*. Unless the Musielak–Orlicz function Φ satisfies the Δ_2 -condition, not all functionals in $(L^\Phi)^*$ are represented by (8) for some $v \in L^{\Phi^*}$. However, every functional $f \in (L^\Phi)^*$ can be uniquely expressed as

$$f = f_c + f_s,$$

where f_c and f_s are said to be the *order continuous* (i.e., $f_c = f_v$ for some $v \in L^{\Phi^*}$) and *singular component* of f , respectively. A functional $f \in (L^\Phi)^*$ for which $f_c = 0$ is called *purely singular*. If the Musielak–Orlicz function Φ is finite-valued, then purely singular functionals are characterized as those functionals vanishing on E^Φ . Unfortunately, this characterization can not be used if the Musielak–Orlicz function Φ is not finite-valued, since we can have $E^\Phi = \{0\}$. We say that a functional $f \in (L^\Phi)^*$ is *positive* if $f(u) \geq 0$ for all non-negative functions $u \in L^\Phi$. To find the order continuous and singular component of a positive functional $f \in (L^\Phi)^*$, we can use

$$f_c(u) = \inf \left\{ \sup_{n \geq 1} f(u_n) : 0 \leq u_n \uparrow u \right\} \quad (9)$$

and

$$f_s(u) = \sup \left\{ \inf_{n \geq 1} f(u_n) : u \geq u_n \downarrow 0 \right\}, \quad (10)$$

for any non-negative functions $u \in L^\Phi$. Expressions (9) and (10) are valid for arbitrary Musielak–Orlicz functions. For any $f \in (L^\Phi)^*$, we define the norms

$$\|f\|_0 = \sup_{u \in L^\Phi} \frac{|f(u)|}{\|u\|_\Phi}, \quad \text{and} \quad \|f\| = \sup_{u \in L_0^\Phi} \frac{|f(u)|}{\|u\|_{\Phi,0}}.$$

Thanks to (9) and (10), we can show, in Section 2, some results related to the norms of functionals in $(L^\Phi)^*$ for arbitrary Musielak–Orlicz functions. Assuming that the Musielak–Orlicz function Φ is finite-valued, one can verify that $(E^\Phi)^* \simeq L^{\Phi^*}$.

Let $(X, \|\cdot\|)$ be a Banach space, whose (topological) dual space is denoted by X^* . A *support functional* at $x \in X \setminus \{0\}$ is a norm-one functional $f \in X^*$ such that $f(x) = \|x\|$. The Hahn–Banach Theorem ensures the existence of at least one support functional. If there exists only one support functional at $x \in X \setminus \{0\}$, then x is said to be a *smooth*

point. A Banach space X is called *smooth* if every $x \in X \setminus \{0\}$ is a smooth point.

The rest of this paper is organized as follows. In Section 2, some results related to the norms of functionals in $(L^\Phi)^*$ are proved for arbitrary Musielak–Orlicz functions. In Section 3, characterization of support functionals and smooth points in L_0^Φ are presented, and we obtain necessary and sufficient conditions for the smoothness of L_0^Φ .

2 Auxiliary results

In this section, some expressions involving the norms of functionals in $(L^\Phi)^*$ are proved for arbitrary Musielak–Orlicz functions. To show these results, we make use of the lemmas below.

Lemma 1. *Let Φ be an arbitrary Musielak–Orlicz function. If $u: T \rightarrow [0, \infty)$ is a measurable function satisfying $\Phi(t, u(t)) < \infty$ for μ -a.e. $t \in T$, then we can find a sequence of non-negative measurable functions $\{u_n\}$ such that $u_n \uparrow u$, and $I_{\Phi^*}(\Phi'_-(t, u_n(t))) < \infty$ for all $n \geq 1$.*

Proof. For each $n \geq 1$, define $\tilde{u}_n(t) = \max(0, u(t) - 1/n)$. In view of $\tilde{u}_n < b_\Phi$, it follows that $\Phi^*(t, \Phi'_-(t, \tilde{u}_n(t))) < \infty$ for μ -a.e. $t \in T$. Let $\{T_n\}$ be a non-decreasing sequence of measurable sets such that $0 < \mu(T_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^\infty T_n) = 0$. We can find, for each $n \geq 1$, a sufficiently large $m_n \geq 1$ such that the set $A_n = \{t \in T_n : \Phi^*(t, \Phi'_-(t, \tilde{u}_n(t))) \leq m_n\}$ satisfies $\mu(T_n \setminus A_n) < 2^{-n}$. Let $B_n = \bigcap_{k=n}^\infty A_k$. Clearly, $B_n \subseteq B_{n+1}$ and $B_n \subseteq T_n$ for all $n \geq 1$. Thus, for any $n \geq m$, we can write

$$\begin{aligned} \mu(T_m \setminus B_n) &= \mu\left(\bigcup_{k=n}^\infty T_m \setminus A_k\right) \leq \sum_{k=n}^\infty \mu(T_m \setminus A_k) \\ &\leq \sum_{k=n}^\infty \mu(T_k \setminus A_k) \leq \sum_{k=n}^\infty 2^{-k} \\ &= 2^{-n+1}, \end{aligned}$$

from which we can conclude that $\mu(T \setminus \bigcup_{n=1}^\infty B_n) = 0$. Defining $u_n = \tilde{u}_n \chi_{B_n}$, we obtain that $u_n \uparrow u$, and $I_{\Phi^*}(\Phi'_-(t, u_n(t))) \leq m_n \mu(T_n) < \infty$ for all $n \geq 1$. \square

Lemma 2. *Let Φ be an arbitrary Musielak–Orlicz function.*

- (a) *Then there exists a non-decreasing sequence of finite-valued Musielak–Orlicz functions $\{\Phi_n\}$ converging upward to Φ , i.e., such that $\Phi_n(t, u) \uparrow \Phi(t, u)$, for all $u \geq 0$, and μ -a.e. $t \in T$.*

- (b) In addition, for any such sequence $\{\Phi_n\}$, if u is a function belonging to L^{Φ_n} for all $n \geq 1$, and the sequence $\{\|u\|_{\Phi_n}\}$ is bounded, then u belongs to L^Φ , and $\|u\|_{\Phi_n} \uparrow \|u\|_\Phi$.

Proof. (a) For each $n \geq 1$, we define the Musielak–Orlicz function

$$\Phi_n(t, u) = \int_0^u \min(\Phi'_-(t, x), n) dx.$$

Clearly, $\Phi_n(t, u) = \Phi(t, u)$ for any $u \geq 0$ satisfying $\Phi'_-(t, u) \leq n$. In addition, $\Phi_n(t, u) \uparrow \infty$ for any $u > 0$ such that $\Phi'_-(t, u) = \infty$. Thus $\Phi_n(t, u) \uparrow \Phi(t, u)$ for all $u \geq 0$.

(b) The case $u = 0$ is trivial. So we assume that $u \neq 0$. Since $I_{\Phi_m}(u/\lambda) \leq I_{\Phi_n}(u/\lambda)$ for any $\lambda > 0$ and $m \leq n$, it follows that $\|u\|_{\Phi_m} \leq \|u\|_{\Phi_n}$. Thus the sequence $\{\|u\|_{\Phi_n}\}$ converges upward to some $c > 0$. In view of Fatou's Lemma, for any $\lambda > c$, we have that

$$I_\Phi(u/\lambda) \leq \liminf_{n \rightarrow \infty} I_{\Phi_n}(u/\lambda) \leq 1.$$

Hence $u \in L^\Phi$ and $\|u\|_\Phi \leq c$. Now, for any $\lambda < c$, and a sufficiently large $n \geq 1$ such that $\|u\|_{\Phi_n} > \lambda$, we obtain that $I_\Phi(u/\lambda) \geq I_{\Phi_n}(u/\lambda) > 1$. Consequently, $\|u\|_\Phi = c$. \square

Proposition 3. *The Orlicz and Luxemburg norms can be expressed respectively as*

$$\|u\|_{\Phi,0} = \sup \left\{ \left| \int_T u v d\mu \right| : v \in L^{\Phi^*} \text{ and } \|v\|_{\Phi^*} \leq 1 \right\} \quad (11)$$

and

$$\|u\|_\Phi = \sup \left\{ \left| \int_T u v d\mu \right| : v \in L^{\Phi^*} \text{ and } \|v\|_{\Phi^*,0} \leq 1 \right\}. \quad (12)$$

Proof. By the definition of Luxemburg norm, it is clear that $v \in L^{\Phi^*}$ satisfies $\|v\|_{\Phi^*} \leq 1$ if and only if $I_{\Phi^*}(v) \leq 1$. Therefore, the Orlicz norm can be expressed as in (11). A consequence of (11) is Hölder's Inequality:

$$\left| \int_T u v d\mu \right| \leq \|u\|_\Phi \|v\|_{\Phi^*,0},$$

which is employed in the proof of (12).

We will show that (12) holds. First, we assume that Φ is finite-valued. Without loss of generality, we also assume that $u \geq 0$ and the supremum in (12) is equal to 1. From Hölder's Inequality, it follows that

$$1 = \sup \left\{ \left| \int_T u v d\mu \right| : v \in L^{\Phi^*} \text{ and } \|v\|_{\Phi^*,0} \leq 1 \right\} \leq \|u\|_\Phi. \quad (13)$$

Suppose that the last inequality in (13) is strict. According to Lemma 1, there exists a sequence of non-negative measurable functions $\{u_n\}$ such that $u_n \uparrow u$ and $I_{\Phi^*}(\Phi'_-(t, u_n(t))) < \infty$ for each $n \geq 1$. Since $\|u\|_{\Phi} > 1$, we obtain that $I_{\Phi}(u) > 1$. Then we can find a sufficiently large $n_0 \geq 1$ for which the function $u_0 := u_{n_0}$ satisfies $I_{\Phi}(u_0) > 1$. Define the function

$$v_0(t) = \frac{\Phi'_-(t, u_0(t))}{1 + I_{\Phi^*}(\Phi'_-(t, u_0(t)))},$$

which belongs to \tilde{L}^{Φ^*} . For any non-negative function $w \in L^{\Phi}$ such that $I_{\Phi}(w) \leq 1$, it follows that

$$\int_T w v_0 d\mu \leq \frac{I_{\Phi}(w) + I_{\Phi^*}(\Phi'_-(t, u_0(t)))}{1 + I_{\Phi^*}(\Phi'_-(t, u_0(t)))} \leq 1.$$

Hence $\|v_0\|_{\Phi^*,0} \leq 1$. In addition, we can write

$$\begin{aligned} \int_T u v_0 d\mu &\geq \int_T u_0 v_0 d\mu \\ &= \frac{I_{\Phi}(u_0) + I_{\Phi^*}(\Phi'_-(t, u_0(t)))}{1 + I_{\Phi^*}(\Phi'_-(t, u_0(t)))} \\ &> 1, \end{aligned}$$

which is a contradiction to (13). Therefore, the last inequality in (13) cannot be strict.

Now assume that Φ is arbitrary. According to Lemma 2-(a), we can find a non-decreasing sequence of finite-valued Musielak–Orlicz functions $\{\Phi_n\}$ converging upward to Φ . The inequality $\Phi^*(t, v) \leq \Phi_n^*(t, v)$, for all $v \geq 0$ and μ -a.e. $t \in T$, implies that $L^{\Phi_n^*} \subseteq L^{\Phi^*}$. Moreover, for any $v \in L^{\Phi_n^*}$,

$$\begin{aligned} \|v\|_{\Phi^*,0} &= \sup \left\{ \left| \int_T u v d\mu \right| : u \in \tilde{L}^{\Phi} \text{ and } I_{\Phi}(u) \leq 1 \right\} \\ &\leq \sup \left\{ \left| \int_T u v d\mu \right| : u \in \tilde{L}^{\Phi_n} \text{ and } I_{\Phi_n}(u) \leq 1 \right\} \\ &= \|v\|_{\Phi_n^*,0}. \end{aligned}$$

Consequently, we can write

$$\begin{aligned} \|u\|_{\Phi_n} &= \sup \left\{ \left| \int_T u v d\mu \right| : v \in L^{\Phi_n^*} \text{ and } \|v\|_{\Phi_n^*,0} \leq 1 \right\} \\ &\leq \sup \left\{ \left| \int_T u v d\mu \right| : v \in L^{\Phi^*} \text{ and } \|v\|_{\Phi^*,0} \leq 1 \right\} \\ &\leq \|u\|_{\Phi}. \end{aligned}$$

In view of Lemma 2-(b), the convergence $\|u\|_{\Phi_n} \uparrow \|u\|_{\Phi}$ implies expression (12). \square

For any $u \in L^\Phi$, we define

$$\theta_\Phi(u) = \inf\{\lambda > 0 : I_\Phi(u/\lambda) < \infty\}$$

and

$$Q_\Phi(u) = \sup\{\inf_{n \geq 1} \|u_n\|_\Phi : |u| \geq u_n \downarrow 0\},$$

$$Q_{\Phi,0}(u) = \sup\{\inf_{n \geq 1} \|u_n\|_{\Phi,0} : |u| \geq u_n \downarrow 0\}.$$

These functionals are related to the norms of purely singular functionals. A remarkable property is that these functionals coincide. To show this claim, we need the following lemma.

Lemma 4. *Let $u \in L^\Phi$ be such that $I_\Phi(u) = \infty$. Then there exists a sequence of measurable functions $\{u_n\}$ such that $|u| \geq u_n \downarrow 0$ and $I_\Phi(u_n) = \infty$ for all $n \geq 1$.*

Proof. Let $B = \{t \in T : \Phi(t, |u(t)|) = \infty\}$. First we assume that $\mu(B) > 0$. Then we can find a non-decreasing sequence of measurable sets $\{B_n\}$, with positive measure $\mu(B_n) > 0$, and such that $B_n \subseteq B$ and $\mu(B_n) \downarrow 0$. For each $n \geq 1$, define the functions $u_n = |u|\chi_{B_n}$. Clearly, $|u| \geq u_n \downarrow 0$ and $I_\Phi(u_n) = \infty$ for all $n \geq 1$. Now suppose that $\mu(B) = 0$. Let $\{T_n\}$ be a non-decreasing sequence of measurable sets such that $0 < \mu(T_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^\infty T_n) = 0$. Define $A_n = \{t \in T_n : \Phi(t, |u(t)|) \leq n\}$. Clearly, the sequence $\{A_n\}$ is non-decreasing, and $\mu(T \setminus \bigcup_{n=1}^\infty A_n) = 0$. For each $n \geq 1$, we define the functions $u_n = |u|\chi_{T \setminus A_n}$, which satisfy $|u| \geq u_n \downarrow 0$. Observing that

$$\infty = I_\Phi(u) = I_\Phi(u_n) + I_\Phi(u\chi_{A_n}) \leq I_\Phi(u_n) + n\mu(T_n),$$

we conclude that $I_\Phi(u_n) = \infty$ for all $n \geq 1$. □

Proposition 5. *For every $u \in L^\Phi$, there holds that $\theta_\Phi(u) = Q_\Phi(u) = Q_{\Phi,0}(u)$.*

Proof. It is clear that $Q_\Phi(u) \leq Q_{\Phi,0}(u)$. Fix any $\varepsilon > 0$. Let $\{u_n\}$ be a sequence in L^Φ such that $|u| \geq u_n \downarrow 0$ and $Q_{\Phi,0}(u) - \varepsilon \leq \inf_{n \geq 1} \|u_n\|_{\Phi,0}$. Take any $\lambda > \theta_\Phi(u)$. In view of $I_\Phi(u/\lambda) < \infty$, we obtain that $I_\Phi(u_n/\lambda) \downarrow 0$. Hence

$$Q_{\Phi,0}(u) - \varepsilon \leq \inf_{n \geq 1} \|u_n\|_{\Phi,0} \leq \inf_{n \geq 1} \lambda(1 + I_\Phi(u_n/\lambda)) = \lambda.$$

Since $\varepsilon > 0$ and $\lambda > \theta_\Phi(u)$ are arbitrary, we have that $Q_{\Phi,0}(u) \leq \theta_\Phi(u)$. If $\theta_\Phi(u) = 0$ then $Q_\Phi(u) = Q_{\Phi,0}(u) = \theta_\Phi(u) = 0$. So we assume that $\theta_\Phi(u) > 0$. Now let $\lambda < \theta_\Phi(u)$. Clearly, $I_\Phi(u/\lambda) = \infty$. According to Lemma 4, we can find a sequence of measurable

functions $\{u_n\}$ such that $|u| \geq u_n \downarrow 0$ and $I_\Phi(u_n/\lambda) = \infty$ for all $n \geq 1$. From the definition of Luxemburg norm, it follows that $\|u_n\|_\Phi \geq \lambda$ for all $n \geq 1$. Then we can write $Q_\Phi(u) \geq \inf_{n \geq 1} \|u_n\|_\Phi \geq \lambda$. Because $\lambda < \theta_\Phi(u)$ is arbitrary, we conclude that $Q_\Phi(u) \geq \theta_\Phi(u)$. Therefore, $\theta_\Phi(u) = Q_\Phi(u) = Q_{\Phi,0}(u)$. \square

Proposition 6. *If the functional $f \in (L^\Phi)^*$ is purely singular, then*

$$\|f\|_0 = \sup_{u \in L^\Phi} \frac{|f(u)|}{Q_\Phi(u)}, \quad \text{and} \quad \|f\| = \sup_{u \in L^\Phi} \frac{|f(u)|}{Q_{\Phi,0}(u)}, \quad (14)$$

or, equivalently,

$$\|f\|_0 = \|f\| = \sup_{u \in \tilde{L}^\Phi} |f(u)| = \sup_{u \in L^\Phi} \frac{|f(u)|}{\theta_\Phi(u)}. \quad (15)$$

Proof. Without loss of generality, we can assume that $f \geq 0$. The equivalence between (14) and (15) follows from Proposition 5. Since $u \in \tilde{L}^\Phi$ if $\|u\|_\Phi \leq 1$, and $\theta_\Phi(u) \leq 1$ for any $u \in \tilde{L}^\Phi$, we can write

$$\begin{aligned} \|f\|_0 &= \sup_{u \in L^\Phi} \frac{|f(u)|}{\|u\|_\Phi} \leq \sup_{u \in \tilde{L}^\Phi} |f(u)| \\ &\leq \sup_{u \in \tilde{L}^\Phi} \frac{|f(u)|}{\theta_\Phi(u)} \leq \sup_{u \in L^\Phi} \frac{|f(u)|}{\theta_\Phi(u)} \\ &= \sup_{u \in L_+^\Phi} \frac{f(u)}{\theta_\Phi(u)}. \end{aligned} \quad (16)$$

Now, for any $u \in L_+^\Phi$, we have that

$$\begin{aligned} f(u) &= \sup\{\inf_{n \geq 1} f(u_n) : u \geq u_n \downarrow 0\} \\ &\leq \sup\{\inf_{n \geq 1} \|f\| \|u_n\|_{\Phi,0} : u \geq u_n \downarrow 0\} \\ &= \|f\| \sup\{\inf_{n \geq 1} \|u_n\|_{\Phi,0} : u \geq u_n \downarrow 0\} \\ &= \|f\| Q_{\Phi,0}(u) = \|f\| \theta_\Phi(u). \end{aligned} \quad (17)$$

From (16) and (17), it follows that $\|f\|_0 \leq \|f\|$. Because $\|f\| \leq \|f\|_0$ is also satisfied, we obtain (15). \square

Proposition 7. *Every functional $f = f_c + f_s \in (L^\Phi)^*$ satisfies $\|f\|_0 = \|f_c\|_0 + \|f_s\|_0$.*

Proof. Since $|f|_c = |f_c|$ and $|f|_s = |f_s|$, we can assume that $f \geq 0$. Clearly, $\|f\|_0 \leq \|f_c\|_0 + \|f_s\|_0$. Given any $\varepsilon > 0$, we select non-negative functions $u, v \in L^\Phi$ with $\|u\|_\Phi \leq 1$ and $\|v\|_\Phi \leq 1$ such that

$$f_c(u) \geq \|f_c\|_0 - \varepsilon, \quad \text{and} \quad f_s(v) \geq \|f_s\|_0 - \varepsilon.$$

In view of (10), there exists a sequence $v \geq v_n \downarrow 0$ satisfying $\inf_{n \geq 1} f_s(v_n) \geq f_s(v) - \varepsilon$. Denote $w_n = \max(u, v_n)$. For any $\eta > 0$, we can find $n_0 \geq 1$ such that $I_\Phi(v_n) \leq \eta$ for every $n \geq n_0$. From

$$I_\Phi(w_n) \leq I_\Phi(u) + I_\Phi(v_n) \leq 1 + \eta,$$

it follows that $\|w_n\|_\Phi \leq 1 + \eta$, for every $n \geq n_0$. Hence, for any $n \geq n_0$, we can write

$$\begin{aligned} (1 + \eta)\|f\|_0 &\geq \|w_n\|_\Phi \|f\|_0 \geq f(w_n) = f_c(w_n) + f_s(w_n) \\ &\geq f_c(u) + f_s(v_n) \geq f_c(u) + f_s(v) - \varepsilon \\ &\geq \|f_c\|_0 + \|f_s\|_0 - 3\varepsilon. \end{aligned}$$

Since $\varepsilon, \eta > 0$ are arbitrary, it follows that $\|f\|_0 \geq \|f_c\|_0 + \|f_s\|_0$. □

Proposition 8. *The norm of any functional $f = f_v + f_s \in (L_0^\Phi)^*$ can be expressed as*

$$\|f\| = \inf\{\lambda > 0 : I_{\Phi^*}(v/\lambda) + \|f_s/\lambda\| \leq 1\}.$$

Proof. Without loss of generality, we assume that $\|f\| = 1$ and $f \geq 0$. Take any $\lambda > 0$ satisfying $I_{\Phi^*}(v/\lambda) + \|f_s/\lambda\| \leq 1$. For any non-negative function $u \in L_0^\Phi$, and arbitrary $k > 0$ such that $I_\Phi(ku) < \infty$, we can write

$$\begin{aligned} \frac{1}{\lambda}|f(u)| &= \frac{1}{k} \left(\int_T (ku)(v/\lambda) d\mu + f_s(ku)/\lambda \right) \\ &\leq \frac{1}{k} (I_\Phi(ku) + I_{\Phi^*}(v/\lambda) + \|f_s/\lambda\|) \\ &\leq \frac{1}{k} (1 + I_\Phi(ku)) \leq \|u\|_{\Phi,0}, \end{aligned}$$

where the first inequality follows from Young's inequality and expression (15). Thus we can conclude that

$$\|f\| \leq \inf\{\lambda > 0 : I_{\Phi^*}(v/\lambda) + \|f_s/\lambda\| \leq 1\}. \quad (18)$$

The function v satisfies the inequality $I_{\Phi^*}(v) \leq 1$. This is a consequence of $\|v\|_{\Phi^*} = \|f_v\| \leq 1$, since $f_v(u) \leq f(u) \leq 1$ for every non-negative function $u \in L_0^\Phi$ such that $\|u\|_{\Phi,0} \leq 1$. According to Lemma 1, there exists a sequence of non-negative measurable functions $\{v_n\}$ such that $v_n \uparrow v$, and $I_\Phi((\Phi^*)'_-(t, v_n(t))) < \infty$ for all $n \geq 1$. Supposing that the first inequality in (18) is strict, we take some $\delta > 0$ such that $I_{\Phi^*}(v) + \|f_s\| > 1 + \delta$. In view of (15), we can find a non-negative function $w \in \tilde{L}^\Phi$ such that $f_s(w) \geq \|f_s\| - \delta/2$. Thus, from (10), there exists a sequence $\{w_n\}$ satisfying $w \geq w_n \downarrow 0$ and $\inf_{n \geq 1} f_s(w_n) \geq f_s(w) - \delta/4$. Select a sufficiently large $n_0 \geq 1$ so that $I_{\Phi^*}(v_{n_0}) \geq$

$I_{\Phi^*}(v) - \delta/8$. Denoting $u_{n_0}(t) = (\Phi^*)'_-(t, v_{n_0}(t))$, we take some $n_1 \geq 1$ for which the function $\tilde{u} = \max(w_{n_1}, u_{n_0})$ satisfies $I_{\Phi}(\tilde{u}) \leq I_{\Phi}(u_{n_0}) + \delta/8$. By these choices, we can write

$$\begin{aligned}
f(\tilde{u}) &= \int_T \tilde{u} v d\mu + f_s(\tilde{u}) \\
&\geq \int_T u_{n_0} v_{n_0} d\mu + f_s(w_{n_1}) \\
&\geq I_{\Phi}(u_{n_0}) + I_{\Phi^*}(v_{n_0}) + f_s(w) - \delta/4 \\
&\geq I_{\Phi}(\tilde{u}) - \delta/8 + I_{\Phi^*}(v) - \delta/8 + \|f_s\| - \delta/2 - \delta/4 \\
&= I_{\Phi}(\tilde{u}) + I_{\Phi^*}(v) + \|f_s\| - \delta \\
&> 1 + I_{\Phi}(\tilde{u}) \\
&\geq \|\tilde{u}\|_{\Phi,0},
\end{aligned}$$

which implies that $\|f\| > 1$. Therefore, the first inequality in (18) cannot be strict. \square

3 Main results

In this section, we provide a characterization of support functionals and smooth points in L_0^{Φ} , and, as a result, we give necessary and sufficient conditions for the smoothness of L_0^{Φ} .

3.1 Support functionals

The characterization of support functionals at a function $u \in L_0^{\Phi}$ depends on whether the set $K(u)$ is empty or not.

Proposition 9. *Let $u \in L_0^{\Phi} \setminus \{0\}$ be such that $K(u) \neq \emptyset$. Then $f = f_v + f_s \in (L_0^{\Phi})^*$ is a support functional at u if and only if, for any $k \in K(u)$,*

- (i) $I_{\Phi^*}(v) + \|f_s\| = 1$,
- (ii) $\|f_s\| = f_s(ku)$, and
- (iii) $\text{sgn } v(t) = \text{sgn } u(t)$ and $|v(t)| \in \partial\Phi(t, |ku(t)|)$ for μ -a.e. $t \in T$.

Proof. Suppose that (i)–(iii) are satisfied. By (i), we have that $\|f\| \leq 1$. For any $k \in K(u)$, we can write

$$\begin{aligned}
f(u) &= \frac{1}{k} \left(\int_T k u v d\mu + f_s(ku) \right) \\
&= \frac{1}{k} (I_{\Phi}(ku) + I_{\Phi^*}(v) + \|f_s\|)
\end{aligned}$$

$$= \frac{1}{k}(1 + I_\Phi(ku)) = \|u\|_{\Phi,0},$$

which implies that $\|f\| = 1$. Therefore, f is a support functional at u . Conversely, let $f = f_v + f_s \in (L_0^\Phi)^*$ be a support functional at u . Using the expression $\frac{1}{k}(1 + I_\Phi(ku)) = \|u\|_{\Phi,0} = f_v(u) + f_s(u)$, for any $k \in K(u)$, we can write

$$\begin{aligned} 1 &= f_v(ku) - I_\Phi(ku) + f_s(ku) \\ &\leq I_\Phi(ku) + I_{\Phi^*}(v) - I_\Phi(ku) + f_s(ku) \\ &= I_{\Phi^*}(v) + f_s(ku) \\ &\leq I_{\Phi^*}(v) + \|f_s\| \leq 1. \end{aligned}$$

Then we obtain (i) and (ii), and $f_v(ku) = I_{\Phi^*}(v) + I_\Phi(ku)$, from which (iii) follows. \square

Proposition 10. *Let $u \in L_0^\Phi \setminus \{0\}$ be such that $K(u) = \emptyset$. Then $f = f_v + f_s \in (L_0^\Phi)^*$ is a support functional at u if and only if*

- (i) $I_{\Phi^*}(v) + \|f_s\| \leq 1$, and
- (ii) $v\chi_{\text{supp } u} = \text{sgn } u \cdot b_{\Phi^*}\chi_{\text{supp } u}$.

Proof. The assumption $K(u) = \emptyset$ implies that $I_\Phi(\lambda u) < \infty$ for all $\lambda > 0$, and $\|u\|_{\Phi,0} = \int_T |u| b_{\Phi^*} d\mu$. It is clear that if (i)–(ii) are satisfied then f is a support functional at u . Conversely, let $f = f_v + f_s \in (L_0^\Phi)^*$ be a support functional at u . Condition (i) follows from Proposition 8. Clearly, $|v| \leq b_{\Phi^*}$. Suppose that the set $\{t \in \text{supp } u : |v(t)| < b_{\Phi^*}(t)\}$ has non-zero measure. From $I_\Phi(\lambda u) < \infty$ for all $\lambda > 0$, we obtain that $f_s(u) = 0$. Then we can write

$$\begin{aligned} f(u) &= f_v(u) + f_s(u) = f_v(u) \\ &= \int_T uv d\mu \leq \int_T |u| |v| d\mu \\ &< \int_T |u| b_{\Phi^*} d\mu = \|u\|_{\Phi,0}, \end{aligned}$$

contradicting the assumption that f is a support functional at u . Therefore, $v\chi_{\text{supp } u} = \text{sgn } u \cdot b_{\Phi^*}\chi_{\text{supp } u}$. \square

Corollary 11. *Let $u \in L_0^\Phi \setminus \{0\}$ for which the set $K(u) \neq \emptyset$ is composed by more than one element. Then there exists only one support functional at u , which is given by f_v with $v(t) = \text{sgn } u(t) \cdot \Phi'_+(t, |k_u^* u(t)|)$.*

Proof. From (ii) in Proposition 9, we conclude that every support functional at u is order continuous. By the definitions of k_u^* and k_u^{**} , it is clear that $I_{\Phi^*}(\Phi'_+(t, |ku(t)|)) = 1$ for

each $k \in [k_u^*, k_u^{**})$. Consequently, $\Phi'_+(t, |k_u^* u(t)|) = \Phi'_-(t, |k_u^{**} u(t)|)$ for μ -a.e. $t \in T$. Therefore, $v(t) = \operatorname{sgn} u(t) \cdot \Phi'_+(t, |k_u^* u(t)|)$ is the unique function satisfying $I_{\Phi^*}(v) = 1$, and such that $\operatorname{sgn} v(t) = \operatorname{sgn} u(t)$ and $|v(t)| \in \partial\Phi(t, |ku(t)|)$ for each $k \in K(u)$. \square

3.2 Smooth points

To find necessary and sufficient conditions for a function in L_0^Φ to be a smooth point, we need some preliminary lemmas. The following result is adapted from [5, Lemma 6] and [11, Lemma 5].

Lemma 12. *If the function $u \in \tilde{L}^\Phi$ satisfies $I_\Phi(\lambda u) = \infty$ for any $\lambda > 1$, then there exist non-increasing sequences of measurable sets $\{A_n\}$ and $\{B_n\}$, converging to the empty set, such that $A_n \cap B_n = \emptyset$ and $I_\Phi(\lambda u \chi_{A_n}) = I_\Phi(\lambda u \chi_{B_n}) = \infty$ for any $\lambda > 1$ and $n \geq 1$.*

Proof. The proof is divided into three cases.

Case 1. Suppose that the measurable set $E = \{t \in T : |u(t)| = b_\Phi(t)\}$ has positive measure $\mu(E) > 0$. Let $\{A_n\}$ and $\{B_n\}$ be non-increasing sequences of measurable sets, converging to the empty set, such that $A_n \cap B_n = \emptyset$ and satisfying $0 < \mu(E \cap A_n)$ and $0 < \mu(E \cap B_n)$. Clearly, for each $n \geq 1$, we have that $I_\Phi(\lambda u \chi_{A_n}) = I_\Phi(\lambda u \chi_{B_n}) = \infty$ for any $\lambda > 1$.

Case 2. Assume that $|u| < b_\Phi$, and for any $\lambda > 1$, the measurable set $F_\lambda = \{t \in T : |\lambda u(t)| > b_\Phi(t)\}$ has positive measure $\mu(F_\lambda) > 0$. Let $\{\lambda_n\}$ be a decreasing sequence in $(1, \infty)$ satisfying $\lambda_n \downarrow 1$. For each $n \geq 1$, denote $F_n = F_{\lambda_n}$. Clearly, $0 < \mu(F_n) \downarrow 0$. For each $n \geq 1$, take disjoint, measurable sets G_n and H_n , whose union is $G_n \cup H_n = F_n \setminus F_{n+1}$, and such that $\mu(G_n) > 0$ and $\mu(H_n) > 0$ if $\mu(F_n \setminus F_{n+1}) > 0$, or $\mu(G_n) = \mu(H_n) = 0$ if $\mu(F_n \setminus F_{n+1}) = 0$. For each $n \geq 1$, define the disjoint sets $A_n = \bigcup_{k=n}^\infty G_k$ and $B_n = \bigcup_{k=n}^\infty H_k$. Clearly, we have that $\mu(A_n) > 0$ and $\mu(B_n) > 0$, for every $n \geq 1$. Take any $\lambda > 1$ and $n \geq 1$. For a sufficiently large $n_0 \geq n$ such that $\lambda \geq \lambda_{n_0}$, it follows that

$$I_\Phi(\lambda u \chi_{A_n}) = \sum_{k=n}^\infty I_\Phi(\lambda u \chi_{G_k}) \geq \sum_{k=n_0}^\infty I_\Phi(\lambda_k u \chi_{G_k}) = \infty.$$

Similarly, we have that $I_\Phi(\lambda u \chi_{B_n}) = \infty$ for any $\lambda > 1$ and $n \geq 1$.

Case 3. Now suppose that $|\bar{\lambda}u| < b_\Phi$ for some $\bar{\lambda} > 1$. Let $\{\lambda_n\}$ be a decreasing sequence in $(1, \bar{\lambda})$ such that $\lambda_n \downarrow 1$. Let $\{T_n\}$ be a non-decreasing sequence of measurable sets such that $0 < \mu(T_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^\infty T_n) = 0$. Define the measurable sets $E_n^m = \{t \in T_m : \Phi(t, |\lambda_n u(t)|) \leq m\}$, for all $n, m \geq 1$. Clearly, $\chi_{E_n^m} \uparrow 1$ as $m \rightarrow \infty$, for each $n \geq 1$. In view of $I_\Phi(\lambda_1 u) = \infty$, we can find $m_1 \geq 1$ such that $F_1 = E_1^{m_1}$ satisfies $2 \leq I_\Phi(\lambda_1 u \chi_{F_1}) \leq m_1 \mu(T_{m_1}) < \infty$. Obviously, $I_\Phi(\lambda_n u \chi_{T \setminus F_1}) = \infty$ for all $n > 1$. Similarly, we can find $m_2 > m_1$ such that defining $F_2 = E_2^{m_2} \cap (T \setminus F_1)$ we get

$F_1 \cap F_2 = \emptyset$ and $2 \leq I_\Phi(\lambda_2 u \chi_{F_2}) \leq m_2 \mu(T_{m_2}) < \infty$. Thus $I_\Phi(\lambda_n u \chi_{T \setminus (F_1 \cup F_2)}) = \infty$ for all $n > 2$. Repeating these steps we obtain a sequence $\{F_n\}$ of pairwise disjoint sets such that $2 \leq I_\Phi(\lambda_n u \chi_{F_n}) < \infty$ for all $n \geq 1$. Since the measure μ is non-atomic, there exist disjoint, measurable sets G_n and H_n , whose union is $F_n = G_n \cup H_n$, such that

$$I_\Phi(\lambda_n u \chi_{G_n}) = I_\Phi(\lambda_n u \chi_{H_n}) = \frac{1}{2} I_\Phi(\lambda_n u \chi_{F_n}) \geq 1.$$

For each $n \geq 1$, define the disjoint sets $A_n = \bigcup_{k=n}^{\infty} G_k$ and $B_n = \bigcup_{k=n}^{\infty} H_k$. Take any $\lambda > 1$ and $n \geq 1$. For some $n_0 \geq n$ such that $\lambda \geq \lambda_{n_0}$, we can write

$$I_\Phi(\lambda u \chi_{A_n}) = \sum_{k=n}^{\infty} I_\Phi(\lambda u \chi_{G_k}) \geq \sum_{k=n_0}^{\infty} I_\Phi(\lambda_k u \chi_{G_k}) = \infty.$$

Analogously, we obtain that $I_\Phi(\lambda u \chi_{B_n}) = \infty$ for any $\lambda > 1$ and $n \geq 1$. \square

Lemma 13. *If the function $u \in \tilde{L}^\Phi$ satisfies $I_\Phi(\lambda u) = \infty$ for any $\lambda > 1$, then there exist two purely singular functionals $s_1 \neq s_2$ in $(L_0^\Phi)^*$, with norms $\|s_1\| = \|s_2\| = 1$, and such that $s_1(u) = s_2(u) = 1$.*

Proof. According to Lemma 12, there exist non-increasing sequences of measurable sets $\{A_n\}$ and $\{B_n\}$, converging to the empty set, such that $A_n \cap B_n = \emptyset$ and $I_\Phi(\lambda u \chi_{A_n}) = I_\Phi(\lambda u \chi_{B_n}) = \infty$ for any $\lambda > 1$, and all $n \geq 1$. Without loss of generality, we assume that $I_\Phi(u \chi_{A_n}) \leq 1$ and $I_\Phi(u \chi_{B_n}) \leq 1$ for all $n \geq 1$. By this assumption, it follows that $\|u \chi_{A_n}\|_\Phi = \|u \chi_{B_n}\|_\Phi = 1$ for all $n \geq 1$. Denote the subspaces

$$\begin{aligned} \mathcal{E}_1 &= \{w \in L^\Phi : \text{supp } w \in T \setminus A_n \text{ for some } n \geq 1\}, \\ \mathcal{E}_2 &= \{w \in L^\Phi : \text{supp } w \in T \setminus B_n \text{ for some } n \geq 1\}. \end{aligned}$$

It is clear that

$$\inf\{\|u - w\|_\Phi : w \in \mathcal{E}_1\} = \inf_{n \geq 1} \|u \chi_{A_n}\|_\Phi = 1.$$

Hence the function u does not belong to the closure of \mathcal{E}_1 . Similarly, u is not in the closure of \mathcal{E}_2 . By the Hahn–Banach Theorem, we can find functionals $s_1, s_2 \in (L_0^\Phi)^*$, with norms $\|s_1\| = \|s_2\| = 1$, and satisfying $s_1(u) = s_2(u) = 1$ and

$$\begin{aligned} s_1(w) &= 0, & \text{for every } w \in \mathcal{E}_1, \\ s_2(w) &= 0, & \text{for every } w \in \mathcal{E}_2. \end{aligned}$$

Since $B_n \subseteq T \setminus A_n$, we have that $s_1(u \chi_{B_n}) = 0$ and $s_2(u \chi_{B_n}) = 1$. Hence $s_1 \neq s_2$. Clearly, the positive and negative parts of s_1 vanish on \mathcal{E}_1 . For any non-negative $w \in L^\Phi$,

it follows that

$$\begin{aligned} ((s_1)_\pm)_c(w) &= \inf\{\sup_{n \geq 1}(s_1)_\pm(w_n) : 0 \leq w_n \uparrow w\} \\ &\leq \sup_{n \geq 1}(s_1)_\pm(w\chi_{T \setminus A_n}) = 0. \end{aligned}$$

Therefore, s_1 is purely singular. Analogously, we have that s_2 is purely singular. \square

Proposition 14. *Let $u \in L_0^\Phi \setminus \{0\}$ be such that $K(u) \neq \emptyset$. Then u is a smooth point if and only if at least one of the following conditions is satisfied:*

- (i) $I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) = 1$, or
- (ii) $I_{\Phi^*}(\Phi'_+(t, |k_u^* u(t)|)) = 1$ and $I_\Phi(\lambda u) < \infty$ for some $\lambda > k_u^*$.

Proof. Assume that $u \in L_0^\Phi \setminus \{0\}$ is a smooth point. If both conditions (i) and (ii) are not satisfied, then at least one of the following expressions holds:

$$I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) < 1 \quad \text{and} \quad I_{\Phi^*}(\Phi'_+(t, |k_u^* u(t)|)) > 1, \quad (19)$$

or

$$I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) < 1 \quad \text{and} \quad I_\Phi(\lambda u) = \infty \text{ for all } \lambda > k_u^*. \quad (20)$$

If (19) is satisfied, then we can find a finite-valued, measurable function v such that $\text{sgn } v(t) = \text{sgn } u(t)$ and $|v(t)| \in \partial\Phi(t, |k_u^* u(t)|)$ for μ -a.e. $t \in T$, and $I_{\Phi^*}(v) > 1$. Denoting $\delta = I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) < 1$, we take any $\eta \in (0, 1 - \delta]$ such that $I_{\Phi^*}(v) \geq 1 + \eta$. Then we can write

$$I_{\Phi^*}(v) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) \geq 1 - \delta + \eta.$$

Because the measure μ is non-atomic, there exists a measurable set E such that

$$I_{\Phi^*}(v\chi_E) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_E(t)) = 1 - \delta + \eta. \quad (21)$$

In view of $\eta \in (0, 1 - \delta]$, we can find disjoint, measurable sets $A, B \subset E$ such that

$$I_{\Phi^*}(v\chi_A) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_A(t)) = \eta, \quad (22)$$

$$I_{\Phi^*}(v\chi_B) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_B(t)) = \eta. \quad (23)$$

Clearly, the intersection of A or B with the set $\{t \in T : v(t) > \Phi'_-(t, |k_u^* u(t)|)\}$ has non-zero measure. Thus the following functions are different:

$$v_1(t) := v(t)\chi_{E \setminus A}(t) + \Phi'_-(t, |k_u^* u(t)|)\chi_{T \setminus (E \setminus A)}(t),$$

$$v_2(t) := v(t)\chi_{E \setminus B}(t) + \Phi'_-(t, |k_u^* u(t)|)\chi_{T \setminus (E \setminus B)}(t).$$

From (21) and (22), we can write

$$\begin{aligned} I_{\Phi^*}(v_1) &= I_{\Phi^*}(v\chi_{E \setminus A}) + I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_{T \setminus (E \setminus A)}(t)) \\ &= I_{\Phi^*}(v\chi_E) - I_{\Phi^*}(v\chi_A) + I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_{E \setminus A}(t)) \\ &= I_{\Phi^*}(v\chi_E) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_E(t)) + \delta - [I_{\Phi^*}(v\chi_A) - I_{\Phi^*}(\Phi'_-(t, |k_u^* u(t)|)\chi_A(t))] \\ &= 1 - \delta + \eta + \delta - \eta = 1. \end{aligned}$$

Analogously, from (21) and (23), it follows that $I_{\Phi_2}(v_2) = 1$. According to Proposition 9, f_{v_1} and f_{v_2} are different support functionals at u .

Now suppose that (20) is satisfied. Let $v(t) = \operatorname{sgn} u(t) \cdot \Phi'_-(t, |k_u^* u(t)|)$. Using Lemma 13, we can find two purely singular functionals $s_1 \neq s_2$ in $(L_0^\Phi)^*$, with norms $\|s_1\| = \|s_2\| = 1 - I_{\Phi^*}(v)$, and such that $s_1(u) = s_2(u) = 1 - I_{\Phi^*}(v)$. According to the proof of Corollary 11, if the set $K(u)$ is composed by more than one element, then $I_{\Phi^*}(v) = 1$, which is a contradiction to (20). Consequently, $K(u) = \{k_u^*\}$. Then we conclude that f_1 and f_2 satisfy conditions (i) and (ii) in Proposition 9, which shows that f_1 and f_2 are different support functionals at u . Therefore, if $u \in L^\Phi \setminus \{0\}$ is a smooth point, then at least one of conditions (i) or (ii) holds.

Next we will show that if at least one of conditions (i) or (ii) is satisfied, then u is a smooth point. Let v be any measurable function such that $\operatorname{sgn} v(t) = \operatorname{sgn} u(t)$ and $|v(t)| \in \partial\Phi(t, |k_u^* u(t)|)$ for μ -a.e. $t \in T$. Assume that (i) is satisfied. It is clear that $I_{\Phi^*}(v) \geq 1$, and that $I_{\Phi^*}(v) = 1$ if and only if $v(t) = v_0(t) = \operatorname{sgn} u(t) \cdot \Phi'_-(t, |k_u^* u(t)|)$. In view of Proposition 9, f_{v_0} is the unique support functional at u . Now suppose that (ii) holds. Then does not exist a non-zero, purely singular functional s such that $\|s\| = s(k_u^* u)$, since in this case we can write that $0 < \|s\| = s(k_u^* u) < s(\lambda u) \leq \|s\|$ for some $\lambda > k_u^*$ such that $I_\Phi(\lambda u) < \infty$. From (ii) in Proposition 9, it follows that every support functional at u is order continuous. Assume that f_v is a support functional at u . Hence $I_{\Phi^*}(v) = 1$. If $k_u^* = k_u^{**}$ then $I_{\Phi^*}(v) = 1$ implies that $v(t) = v_1(t) = \operatorname{sgn} u(t) \cdot \Phi'_+(t, |k_u^* u(t)|)$. In view of Corollary 11, if the set $K(u)$ is composed by more than one element, then u is a smooth point and $v(t) = v_1(t) = \operatorname{sgn} u(t) \cdot \Phi'_+(t, |k_u^* u(t)|)$. Thus, assuming that (ii) is satisfied, we have that f_{v_1} is the unique support functional at u . \square

Proposition 15. *Let $u \in L_0^\Phi \setminus \{0\}$ be such that $K(u) = \emptyset$. Then u is a smooth point if and only if*

- (i) $I_{\Phi^*}(b_{\Phi^*}\chi_{\operatorname{supp} u}) = 1$ and $a_{\Phi^*}\chi_{T \setminus \operatorname{supp} u} = 0$, or
- (ii) $I_{\Phi^*}(b_{\Phi^*}) < 1$ and $\mu(T \setminus \operatorname{supp} u) = 0$.

Proof. Assume that u is a smooth point. Let $f = f_v + f_s \in (L_0^\Phi)^*$ be a support functional at u . According to Proposition 10, we have that $I_{\Phi^*}(v) + \|f_s\| \leq 1$ and $v\chi_{\text{supp } u} = \text{sgn } u \cdot b_{\Phi^*}\chi_{\text{supp } u}$. Suppose that Φ does not satisfy the Δ_2 -condition. If $I_{\Phi^*}(v) < 1$ then any functional $g = f_v + s$ such that $I_{\Phi^*}(v) + \|s\| \leq 1$, where s is a purely singular functional, is a support functional at u . Consequently, $I_{\Phi^*}(v) = 1$. We cannot have $I_{\Phi^*}(b_{\Phi^*}\chi_{\text{supp } u}) < 1$, since f_w with $w = \text{sgn } u \cdot b_{\Phi^*}\chi_{\text{supp } u} \neq v$ would be a support functional at u . By $I_{\Phi^*}(v\chi_{T \setminus \text{supp } u}) = 0$, it follows that $0 \leq |v|\chi_{T \setminus \text{supp } u} \leq a_{\Phi^*}\chi_{T \setminus \text{supp } u}$. If $a_{\Phi^*}\chi_{T \setminus \text{supp } u} > 0$ then it is clear that f_w with $w = v\chi_{\text{supp } u} + \frac{1}{2}(a_{\Phi^*} - |v|)\chi_{T \setminus \text{supp } u} \neq v$ is a support functional at u . Hence $a_{\Phi^*}\chi_{T \setminus \text{supp } u} = 0$. Therefore, (i) holds. Now suppose that Φ satisfies the Δ_2 -condition. Thus $f_s = 0$. If $I_{\Phi^*}(b_{\Phi^*}\chi_{\text{supp } u}) = 1$ then proceeding as above we obtain that $a_{\Phi^*}\chi_{T \setminus \text{supp } u} = 0$. Assume that $I_{\Phi^*}(b_{\Phi^*}\chi_{\text{supp } u}) < 1$. If $\mu(T \setminus \text{supp } u) \neq 0$ then it is clear that we can find $w \in \tilde{L}^\Phi$ with $w \neq v$ and $I_{\Phi^*}(w) \leq 1$. Hence f_w is a support functional at u . Consequently, $\mu(T \setminus \text{supp } u) = 0$. Then (i) or (ii) is satisfied.

Conversely, if (i) holds then in view of Proposition 10 it is clear that f_v with $v = b_{\Phi^*}\chi_{\text{supp } u}$ is the unique support functional at u . Assume that (ii) is satisfied. For any $w \in L^\Phi$, and $\lambda > 0$, we have that

$$\begin{aligned} I_\Phi(\lambda w) &\leq I_\Phi(\lambda w) + I_{\Phi^*}(\Phi'_-(t, |\lambda w(t)|)) \\ &= \int_T |\lambda w(t)| \Phi'_-(t, |\lambda w(t)|) d\mu \\ &\leq \int_T |\lambda w| b_{\Phi^*} d\mu \\ &= \|\lambda w\|_{\Phi, 0} < \infty. \end{aligned}$$

Then Φ satisfies the Δ_2 -condition, and any functional in $(L^\Phi)^*$ is order continuous. Consequently, f_v with $v = b_{\Phi^*}$ is the unique support functional at u . \square

3.3 Smoothness of L_0^Φ

Below we state the main result of this paper, which provides necessary and sufficient conditions for the smoothness of L_0^Φ .

Proposition 16. *The Musielak–Orlicz space L_0^Φ is smooth if and only if*

- (a) $\Phi^*(t, b_{\Phi^*}(t)) = \infty$ for μ -a.e. $t \in T$,
- (b) Φ satisfies the Δ_2 -condition, and
- (c) $\Phi(t, \cdot)$ is continuously differentiable (with $\Phi'_+(t, 0) = 0$) for μ -a.e. $t \in T$.

The proof of this proposition requires some preliminary results and observations. Let $u \in L_0^\Phi \setminus \{0\}$ be such that $K(u) \neq \emptyset$. Suppose that $f = f_v + f_s$ is a support functional of u with non-zero singular component $f_s \neq 0$. From condition (ii) in Proposition 9, we have that $\|f_s\| = \sup_{u \in \tilde{L}^\Phi} |f(u)| = f_s(ku)$ for any $k \in K(u)$. This result implies that $K(u) = \{k\}$, and $I_\Phi(\lambda u) = \infty$ for any $\lambda > k$. Moreover, in view of (ii) in Proposition 9, it follows that $I_{\Phi^*}(v) = 1 - \|f_s\| < 1$. Thus the existence of a function $u \in L_0^\Phi$ such that $I_\Phi(\lambda u) = \infty$ for any $\lambda > k$, and $I_{\Phi^*}(\Phi'_-(t, |ku(t)|)) < 1$, is a necessary condition for the existence of a support functional with non-zero singular component. Thanks to the result below, we can prove Proposition 16 without using the Bishop–Phelps Theorem (cf. [7, Theorem 2.3]).

Proposition 17. *Let Φ be a Musielak–Orlicz function not satisfying the Δ_2 -condition. Then there exists a function $u \in \tilde{L}^\Phi$ such that $I_\Phi(\lambda u) = \infty$ for any $\lambda > 1$, and $I_{\Phi^*}(\Phi'_+(t, |u(t)|)) < 1$.*

Notice that, for the function $u \in \tilde{L}^\Phi$ in the proposition above, we have that $K(u) = \{1\}$. To show Proposition 17, we make use of the lemma below, which is stated without proof (see [8, Lemma 8.3]).

Lemma 18. *Let μ be a non-atomic, σ -finite measure. If $\{\alpha_n\}$ is a sequence of positive, real numbers, and $\{u_n\}$ is a sequence of finite-valued, non-negative, measurable functions, such that*

$$\int_T u_n d\mu \geq 2^n \alpha_n, \quad \text{for all } n \geq 1,$$

then there exist an increasing sequence $\{n_i\}$ of natural numbers and a sequence $\{A_i\}$ of pairwise disjoint, measurable sets such that

$$\int_{A_i} u_{n_i} d\mu = \alpha_{n_i}, \quad \text{for all } i \geq 1.$$

One can easily verify that the Δ_2 -condition given by (7) is equivalent to the existence of a constant $\alpha > 0$, and a non-negative function $f \in \tilde{L}^\Phi$ such that

$$\alpha \Phi(t, u) \leq \Phi(t, \tfrac{1}{2}u), \quad \text{for all } u > f(t). \quad (24)$$

Moreover, a Musielak–Orlicz function Φ satisfies the Δ_2 -condition if, and only if, for every $\lambda \in (0, 1)$, there exist a constant $\alpha_\lambda \in (0, 1)$, and a non-negative function $f_\lambda \in \tilde{L}^\Phi$ such that

$$\alpha_\lambda \Phi(t, u) \leq \Phi(t, \lambda u), \quad \text{for all } u > f_\lambda(t). \quad (25)$$

We will use this observation to prove the next result.

Lemma 19. *Let Φ be a Musielak–Orlicz function not satisfying the Δ_2 -condition. Assume that $\Phi(t, b_\Phi(t)) = \infty$ for μ -a.e. $t \in T$. Then we can find a strictly increasing sequence $\{\lambda_n\}$ in $(0, 1)$ converging upward to 1, and sequences $\{u_n\}$ and $\{A_n\}$ of finite-valued, measurable functions, and pairwise disjoint, measurable sets, respectively, such that*

$$I_\Phi(u_n \chi_{A_n}) = 1 \quad \text{and} \quad I_\Phi(\lambda_n u_n \chi_{A_n}) \leq 2^{-n}, \quad \text{for all } n \geq 1. \quad (26)$$

Proof. Because the Musielak–Orlicz function Φ does not satisfy the Δ_2 -condition, for any $\lambda \in (0, 1)$, there do not exist a constant $\alpha \in (0, 1)$ and a non-negative function $f \in \tilde{L}^\Phi$ such that

$$\alpha \Phi(t, u) \leq \Phi(t, \lambda u), \quad \text{for all } u > f(t). \quad (27)$$

Let $\{\lambda'_m\}$ be a strictly increasing sequence in $(0, 1)$ such that $\lambda'_m \uparrow 1$. Define the non-negative functions

$$f_m(t) = \sup\{u \in (0, b_\Phi(t)) : 2^{-m} \Phi(t, u) > \Phi(t, \lambda'_m u)\}, \quad \text{for all } m \geq 1,$$

where we adopt the convention that $\sup \emptyset = 0$. Since (27) is not satisfied, we have that $I_\Phi(f_m) = \infty$ for each $m \geq 1$. For every rational number $r > 0$, define the measurable sets

$$A_{m,r} = \{t \in T : r \in (0, b_\Phi(t)) \text{ and } 2^{-m} \Phi(t, u) > \Phi(t, \lambda'_m u)\}$$

and the simple functions $u_{m,r} = r \chi_{A_{m,r}}$. For $r = 0$, set $u_{m,r} = 0$. Let $\{r_i\}$ be an enumeration of the non-negative rational numbers with $r_1 = 0$. Define the non-negative, simple functions $v_{m,k} = \max_{1 \leq i \leq k} u_{m,r_i}$, for each $m, k \geq 1$. By the left-continuity of $\Phi(t, \cdot)$, it follows that $v_{m,k} \uparrow f_m$ as $k \rightarrow \infty$. In virtue of the Monotone Convergence Theorem, for each $m \geq 1$, we can find some $k_m \geq 1$ such that the function $v_m = v_{m,k_m}$ satisfies $I_\Phi(v_m) \geq 2^m$. Clearly, we have that $\Phi(t, v_m(t)) < \infty$ and $2^{-m} \Phi(t, v_m(t)) \geq \Phi(t, \lambda'_m v_m(t))$. By Lemma 18, there exist an increasing sequence $\{m_n\}$ of indices and a sequence $\{A_n\}$ of pairwise disjoint, measurable sets such that $I_\Phi(v_{m_n} \chi_{A_n}) = 1$. Taking $\lambda_n = \lambda'_{m_n}$, $u_n = v_{m_n}$ and A_n , we obtain (26). \square

Proof of Proposition 17. Suppose that the measurable set $E = \{t \in T : \Phi(t, b_\Phi(t)) < \infty\}$ has positive measure $\mu(E) > 0$. Take a measurable set $F \subseteq E$ such that $\mu(F) > 0$ and $I_\Phi(b_\Phi \chi_F) < \infty$. Since the measure μ is non-atomic, we can find pairwise disjoint, measurable sets $A_n \subset F$ such that $\mu(A_n) > 0$ and $F = \bigcup_{n=1}^{\infty} A_n$. Let $\{\lambda_n\}$ be a strictly increasing sequence in $(0, 1)$ such that $\lambda_n \uparrow 1$. For each $n \geq 1$, select a measurable set $B_n \subseteq A_n$ such that $\mu(B_n) > 0$ and

$$I_{\Phi^*}(\Phi'_+(t, \lambda_n b_\Phi(t) \chi_{B_n}(t))) < 2^{-n}. \quad (28)$$

Define the function $u = \sum_{n=1}^{\infty} \lambda_n b_{\Phi} \chi_{B_n}$. Clearly, $I_{\Phi}(u) < \infty$. For any $\lambda > 1$, and some $n_0 \geq 1$ such that $\lambda \lambda_{n_0} > 1$, we have that

$$I_{\Phi}(\lambda u) = \sum_{n=1}^{\infty} I_{\Phi}(\lambda \lambda_n b_{\Phi} \chi_{B_n}) \geq I_{\Phi}(\lambda \lambda_{n_0} b_{\Phi} \chi_{B_{n_0}}) = \infty.$$

By (28), it follows that

$$I_{\Phi^*}(\Phi'_+(t, u(t))) = \sum_{n=1}^{\infty} I_{\Phi^*}(\Phi'_+(t, \lambda_n b_{\Phi}(t) \chi_{B_n}(t))) < \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Now assume that $\Phi(t, b_{\Phi}(t)) = \infty$ for μ -a.e. $t \in T$. Let $\{\lambda_n\}$, $\{u_n\}$ and $\{A_n\}$ be the sequences in the statement of Lemma 19. For a sufficiently large natural number $n_0 > 1$ such that $\lambda_{n_0} > 1/n_0$ and $\sum_{n=n_0}^{\infty} n 2^{-n} < 1$, we define the function $u = \sum_{n=n_0}^{\infty} \lambda'_n u_n \chi_{A_n}$, where $\lambda'_n = \lambda_n - 1/n$, for each $n \geq n_0$. Then we can write

$$I_{\Phi}(u) = \sum_{n=n_0}^{\infty} I_{\Phi}(\lambda'_n u_n \chi_{A_n}) \leq \sum_{n=n_0}^{\infty} I_{\Phi}(\lambda_n u_n \chi_{A_n}) \leq \sum_{n=n_0}^{\infty} 2^{-n} < \infty.$$

Given any $\lambda > 1$, we take some $n_1 \geq n_0$ satisfying $\lambda \lambda'_{n_1} \geq 1$, so that

$$I_{\Phi}(\lambda u) = \sum_{n=n_0}^{\infty} I_{\Phi}(\lambda \lambda'_n u_n \chi_{A_n}) \geq \sum_{n=n_1}^{\infty} I_{\Phi}(u_n \chi_{A_n}) = \infty.$$

For each $n \geq n_0$, we obtain that

$$\begin{aligned} I_{\Phi^*}(\Phi'_+(t, \lambda'_n u_n(t) \chi_{A_n}(t))) &\leq I_{\Phi}(\lambda'_n u_n \chi_{A_n}) + I_{\Phi^*}(\Phi'_+(t, \lambda'_n u_n(t) \chi_{A_n}(t))) \\ &= \int_{A_n} \lambda'_n u_n(t) \Phi'_+(t, \lambda'_n u_n(t)) d\mu \\ &\leq \lambda'_n \frac{1}{\lambda_n - \lambda'_n} [I_{\Phi}(\lambda_n u_n \chi_{A_n}) - I_{\Phi}(\lambda'_n u_n \chi_{A_n})] \\ &\leq \frac{1}{\lambda_n - \lambda'_n} I_{\Phi}(\lambda_n u_n \chi_{A_n}) \\ &\leq n 2^{-n}. \end{aligned}$$

Hence it follows that

$$I_{\Phi^*}(\Phi'_+(t, u(t))) = \sum_{n=n_0}^{\infty} I_{\Phi^*}(\Phi'_+(t, \lambda'_n u_n(t) \chi_{A_n}(t))) \leq \sum_{n=n_0}^{\infty} n 2^{-n} < 1,$$

which completes the proof. \square

Remark 20. Let Φ be a Musielak–Orlicz function not satisfying the Δ_2 -condition and such that $\Phi(t, b_\Phi(t)) = \infty$ for μ -a.e. $t \in T$. Then we can find functions u_* and u^* in L^Φ such that

$$\begin{cases} I_\Phi(\lambda u_*) < \infty, & \text{for } 0 \leq \lambda \leq 1, \\ I_\Phi(\lambda u_*) = \infty, & \text{for } 1 < \lambda, \end{cases} \quad (29)$$

and

$$\begin{cases} I_\Phi(\lambda u^*) < \infty, & \text{for } 0 \leq \lambda < 1, \\ I_\Phi(\lambda u^*) = \infty, & \text{for } 1 \leq \lambda. \end{cases} \quad (30)$$

We construct these functions using the sequences $\{\lambda_n\}$, $\{u_n\}$ and $\{A_n\}$ in Lemma 19. Define $u_* = \sum_{n=1}^{\infty} \lambda_n u_n \chi_{A_n}$ and $u^* = \sum_{n=1}^{\infty} u_n \chi_{A_n}$. For any $0 \leq \lambda \leq 1$, we have that

$$I_\Phi(\lambda u_*) \leq I_\Phi(u_*) = \sum_{n=1}^{\infty} I_\Phi(\lambda_n u_n \chi_{A_n}) \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

For any $\lambda > 1$, take a natural number $n_0 \geq 1$ such that $\lambda \lambda_{n_0} \geq 1$. Then we can write

$$I_\Phi(\lambda u_*) = \sum_{n=1}^{\infty} I_\Phi(\lambda \lambda_n u_n \chi_{A_n}) \geq \sum_{n=n_0}^{\infty} I_\Phi(u_n \chi_{A_n}) = \infty.$$

With respect to u^* , it is clear that $I_\Phi(\lambda u^*) = \infty$ for any $\lambda \geq 1$. If $\lambda < 1$ and the natural number $n_0 \geq 1$ is such that $\lambda \leq \lambda_{n_0}$, we obtain that

$$I_\Phi(\lambda u^*) = \sum_{n=1}^{\infty} I_\Phi(\lambda u_n \chi_{A_n}) \leq \sum_{n=1}^{n_0-1} I_\Phi(\lambda u_n \chi_{A_n}) + \sum_{n=n_0}^{\infty} I_\Phi(\lambda_n u_n \chi_{A_n}) < \infty.$$

Thus the functions u_* and u^* satisfy (29) and (30).

Thanks to the lemma below, we avoid the use of measurable selectors in the proof of Proposition 16 (cf. [7, Theorem 2.2]).

Lemma 21. *Let Φ be a finite-valued Musielak–Orlicz function. For any $\delta > 0$, the function*

$$u_\delta(t) = \sup\{u \geq 0 : \Phi'_+(t, x) - \Phi'_-(t, x) < \delta \text{ for all } 0 \leq x \leq u\}$$

is measurable (where we adopted the convention $\Phi'_-(t, 0) = 0$). Moreover, denoting $H = \{t \in T : u_\delta(t) < \infty\}$, then $\Phi'_+(t, u_\delta(t)) - \Phi'_-(t, u_\delta(t)) \geq \delta$ for μ -a.e. $t \in H$.

Proof. Without loss of generality, we assume that, for each $t \in T$, the function $\Phi(t, \cdot)$ satisfies conditions (i) and (ii) in the definition of Musielak–Orlicz functions. For any

$\varepsilon > 0$, we define the function

$$u_{\delta,\varepsilon}(t) = \sup\{u \geq 0 : \Phi'_-(t, x + \varepsilon) - \Phi'_-(t, x) < \delta \text{ for all } 0 \leq x \leq u\}.$$

We will verify that $u_{\delta,\varepsilon}$ is measurable. Fixed any $u \geq 0$, denote $A_u = \{t \in T : \Phi'_-(t, x + \varepsilon) - \Phi'_-(t, x) < \delta \text{ for all } 0 \leq x \leq u\}$. Let $\{\delta_n\}$ be a sequence in $(0, \delta)$ such that $\delta_n \uparrow \delta$. Now define $\tilde{A}_u = \bigcup_{n=1}^{\infty} \bigcap_{r \in [0, u] \cap \mathbb{Q}} B_r^n$, where $B_r^n = \{t \in T : \Phi'_-(t, r + \varepsilon) - \Phi'_-(t, r) < \delta_n\}$, for each $r \in [0, u] \cap \mathbb{Q}$ and $n \geq 1$. Clearly, the sets B_r^n are measurable, and then \tilde{A}_u is also measurable. We will show that A_u and \tilde{A}_u coincide. It is clear that $A_u \subseteq \tilde{A}_u$. Fix any $t \in \tilde{A}_u$. Hence $t \in \bigcap_{r \in [0, u] \cap \mathbb{Q}} B_r^{n_0}$ for some $n_0 \geq 1$. For any $x \in (0, u]$, take a sequence of rational numbers $\{r_k\}$ in $(0, x)$ such that $r_k \uparrow x$. Since $\Phi'_-(t, r_k + \varepsilon) - \Phi'_-(t, r_k) < \delta_{n_0} < \delta$ for all $k \geq 1$, it follows that $\Phi'_-(t, x + \varepsilon) - \Phi'_-(t, x) \leq \delta_{n_0} < \delta$. Thus $t \in A_u$, which implies that $A_u = \tilde{A}_u$. Consequently, A_u is measurable. Let $\{r_i\}$ be an enumeration of the non-negative, rational numbers with $r_1 = 0$. Clearly, the non-negative simple functions $\bar{u}_n = \max_{1 \leq i \leq n} r_i \chi_{A_{r_i}}$ converge upward to $u_{\delta,\varepsilon}$. Thus the function $u_{\delta,\varepsilon}$ is measurable.

If we show that $u_{\delta,\varepsilon} \uparrow u_\delta$ as $\varepsilon \downarrow 0$, then u_δ is measurable. It is clear that $u_{\delta,\varepsilon_2}(t) \leq u_{\delta,\varepsilon_1}(t) \leq u_\delta(t)$ for any $0 < \varepsilon_1 \leq \varepsilon_2$. Hence $u_{\delta,\varepsilon}(t) \uparrow c(t)$ for some measurable function c . Let $t \in T$ be such that $c(t) = 0$. In view of $u_{\delta,\varepsilon}(t) \leq c(t) = 0$ for all $\varepsilon > 0$, we obtain that $\Phi'_-(t, 0 + \varepsilon) - \Phi'_-(t, 0) \geq \delta$ for all $\varepsilon > 0$. Thus $\Phi'_+(t, 0) - \Phi'_-(t, 0) \geq \delta$, which implies that $u_\delta(t) = 0$. Now suppose that there exists some $t \in T$ satisfying $0 < c(t) < u_\delta(t)$. Clearly, $\Phi'_+(t, c(t)) - \Phi'_-(t, c(t)) < \delta$. Then we can find $\eta > 0$ so that

$$\Phi'_-(t, x_2) - \Phi'_-(t, x_1) < \delta, \quad \text{for all } x_1 \in (c(t) - \eta, c(t)], x_2 \in (c(t), c(t) + \eta). \quad (31)$$

Let $\varepsilon \in (0, \eta/2)$ be such that $u_{\delta,\varepsilon}(t) \in (c(t) - \eta/2, c(t)]$. By the definition of $u_{\delta,\varepsilon}(t)$, there exists $x_0 \in [u_{\delta,\varepsilon}(t), u_{\delta,\varepsilon}(t) + \eta/2]$ with $\Phi'_-(t, x_0 + \varepsilon) - \Phi'_-(t, x_0) \geq \delta$. Denoting $x_1 = \min(x_0, c(t))$ and $x_2 = x_0 + \varepsilon$, we can write $\Phi'_-(t, x_2) - \Phi'_-(t, x_1) \geq \Phi'_-(t, x_0 + \varepsilon) - \Phi'_-(t, x_0) \geq \delta$. This is a contradiction to (31), since $x_1 \in (c(t) - \eta, c(t)]$ and $x_2 \in (c(t), c(t) + \eta)$. Therefore, $u_{\delta,\varepsilon} \uparrow u_\delta$ as $\varepsilon \downarrow 0$.

Assume that $\Phi'_+(t, u_\delta(t)) - \Phi'_-(t, u_\delta(t)) < \delta$ for all $t \in E$, for some measurable set $E \subseteq H$, with non-zero measure $\mu(E) > 0$. Then we can find $\varepsilon > 0$ and measurable set $F \subseteq E$, with non-zero measure $\mu(F) > 0$, such that $\Phi'_-(t, x) - \Phi'_-(t, u_\delta(t)) < \delta$ for all $x \in (u_\delta(t), u_\delta(t) + \varepsilon)$ and all $t \in F$. Fixed any $t \in F$ and $x \in (u_\delta(t), u_\delta(t) + \varepsilon)$, select some $y \in (x, u_\delta(t) + \varepsilon)$. Then we can write that $\Phi'_+(t, x) - \Phi'_-(t, x) \leq \Phi'_-(t, y) - \Phi'_-(t, u_\delta(t)) < \delta$. This result is a contradiction to the definition of u_δ . Consequently, $\Phi'_+(t, u_\delta(t)) - \Phi'_-(t, u_\delta(t)) \geq \delta$ for μ -a.e. $t \in H$. \square

Finally, we can prove the result stated in the beginning of this section.

Proof of Proposition 16. Assume that (a)–(c) are satisfied. Let $u \in L_0^\Phi \setminus \{0\}$. For any $k > 0$ and $\lambda > 1$, we can write

$$\begin{aligned}\Phi^*(t, \Phi'_+(t, ku)) &\leq \Phi(t, ku) + \Phi^*(t, \Phi'_+(t, ku)) = ku\Phi'_+(t, ku) \\ &\leq \frac{1}{\lambda - 1} \int_{ku}^{\lambda ku} \Phi'_+(t, x) dx \leq \frac{1}{\lambda - 1} \Phi(t, \lambda ku).\end{aligned}$$

From (b), we obtain that $I_{\Phi^*}(\Phi'_+(t, |ku(t)|)) < \infty$ for any $k > 0$. Since $\Phi'_+(t, 0) = 0$ and $\Phi^*(t, b_{\Phi^*}(t)) = \infty$ for μ -a.e. $t \in T$, it follows that $I_{\Phi^*}(\Phi'_+(t, |ku(t)|)) \downarrow 0$ as $k \downarrow 0$, and $I_{\Phi^*}(\Phi'_+(t, |ku(t)|)) \uparrow \infty$ as $k \uparrow \infty$. By the continuity of $\Phi'_+(t, \cdot)$, there exists only one measurable function v satisfying $I_{\Phi^*}(v) = 1$ and such that $\text{sgn } v(t) = \text{sgn } u(t)$ and $|v(t)| \in \partial\Phi(t, |ku(t)|)$ for μ -a.e. $t \in T$, for all $k \in K(u)$. According to Proposition 14, the function u is a smooth point.

Conversely, let L^Φ be a smooth Musielak–Orlicz space. If $E = \{t \in T : \Phi^*(t, b_{\Phi^*}(t)) < \infty\}$ has non-zero measure, then we can find a measurable set $F \subseteq E$, with $\chi_F \in \tilde{L}^\Phi \setminus \{0\}$, and such that $I_{\Phi^*}(b_{\Phi^*}\chi_F) < 1$ and $\mu(T \setminus F) > 0$. In view of Proposition 15, the function χ_F is not a smooth point. This result shows that $\Phi^*(t, b_{\Phi^*}(t)) = \infty$ for μ -a.e. $t \in T$. Assume that Φ does not satisfy the Δ_2 -condition. According to Proposition 17, there exists a function $u \in \tilde{L}^\Phi$ such that $I_\Phi(\lambda u) = \infty$ for any $\lambda > 1$, and $I_{\Phi^*}(\Phi'_+(t, |u(t)|)) < 1$. Clearly, $K(u) = \{1\}$. By Proposition 14, we obtain that u is not a smooth point. Thus Φ satisfies the Δ_2 -condition.

Now suppose that $\Phi(t, \cdot)$ is not continuously differentiable for μ -a.e. $t \in T$. According to Lemma 21, for any $\delta > 0$, the function

$$u_\delta(t) = \sup\{u \geq 0 : \Phi'_+(t, x) - \Phi'_-(t, x) < \delta \text{ for all } 0 \leq x \leq u\}$$

is measurable. From the assumption that $\Phi(t, \cdot)$ is not continuously differentiable for μ -a.e. $t \in T$, we can find some $\delta_0 > 0$ for which the measurable set $H = \{t \in T : u_{\delta_0}(t) < \infty\}$ has non-zero measure. Denote $u = u_{\delta_0}$. In view of Lemma 21, we have that $\Phi'_+(t, u(t)) - \Phi'_-(t, u(t)) \geq \delta$, for μ -a.e. $t \in H$. Let $A \subseteq H$ be a measurable set, with non-zero measure, such that $T \setminus A$ has non-zero measure, and $I_\Phi(u\chi_A) < \infty$ and $I_{\Phi^*}(\Phi'_+(t, u(t)\chi_A(t))) \leq 1$. We take disjoint, measurable sets E and F , with non-zero measure, satisfying $A = E \cup F$ and

$$\int_E [\Phi^*(\Phi'_+(t, u(t))) - \Phi^*(\Phi'_-(t, u(t)))] d\mu = \int_F [\Phi^*(\Phi'_+(t, u(t))) - \Phi^*(\Phi'_-(t, u(t)))] d\mu,$$

Thus we can write

$$I_{\Phi^*}(\Phi'_+(t, u(t))\chi_E(t)) + I_{\Phi^*}(\Phi'_-(t, u(t))\chi_F(t))$$

$$= I_{\Phi^*}(\Phi'_+(t, u(t))\chi_E(t)) + I_{\Phi^*}(\Phi'_-(t, u(t))\chi_F(t)) = c \leq 1.$$

Since the set $T \setminus A$ has non-zero measure, and $\Phi^*(t, b_{\Phi^*}(t)) = \infty$ for μ -a.e. $t \in T$, we can find a sufficiently large $n_0 \geq 1$ for which $I_{\Phi^*}(\Phi'_+(t, n_0)\chi_{T \setminus A}(t)) \geq 1 - c$. Let $B \subseteq T \setminus A$ be a measurable set for which $I_{\Phi^*}(\Phi'_+(t, n_0)\chi_B) = 1 - c$. Define the functions

$$\begin{aligned} v_1 &= \Phi'_+(t, u(t))\chi_E(t) + \Phi'_-(t, u(t))\chi_F(t) + \Phi'_+(t, n_0)\chi_B(t), \\ v_2 &= \Phi'_-(t, u(t))\chi_E(t) + \Phi'_+(t, u(t))\chi_F(t) + \Phi'_+(t, n_0)\chi_B(t). \end{aligned}$$

By $I_{\Phi^*}(v_1) = I_{\Phi^*}(v_2) = 1$, it follows that $\|v_1\|_{\Phi^*} = \|v_2\|_{\Phi^*} = 1$. Now define $\tilde{u} = u\chi_A + n_0\chi_B$. Clearly, $|f_{v_i}(\tilde{u})| \leq \|v_i\|_{\Phi^*}\|\tilde{u}\|_{\Phi,0} = \|\tilde{u}\|_{\Phi,0}$. In addition,

$$\|\tilde{u}\|_{\Phi,0} \leq I_{\Phi}(\tilde{u}) + 1 = I_{\Phi}(\tilde{u}) + I_{\Phi^*}(v_i) = \int_T \tilde{u}v_i d\mu = f_{v_i}(\tilde{u}),$$

which implies that $f_{v_i}(\tilde{u}) = \|\tilde{u}\|_{\Phi,0}$. Thus f_{v_1} and f_{v_2} are different support functionals at \tilde{u} . This contradiction shows that $\Phi(t, \cdot)$ is continuously differentiable for μ -a.e. $t \in T$. \square

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